The Fundamental Theorem of Geometric Calculus

Jan 3rd 2026

Something I have been wanting to understand better for a while. Prompted by Joseph sharing insights about analytic complex functions. Following Doran & Lasenby Chapter 6.

In a nutshell

The fundamental theorem of calculus (1-d version)

$$\int_a^b \frac{df(x)}{dx}\, dx = \underbrace{f(b) - f(a)}$$

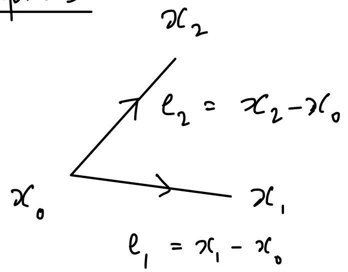
integral of derivative of a function (over some domain) = the function integrated over the domain's boundary.

Many versions of this statement exist in higher dimensions:

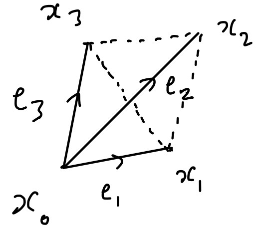
1. Green's Theorem
2. The Divergence Theorem
3. Stokes' Theorem
4. Cauchy's Integral Formula
5. Generalized Stokes.

That 1, 2 and 3 are special cases of 5 is well-known. That 4 is also, is less so. FTGC is the Geometric Algebra version of 5.

Simplices



$n = 2$



$n = 3$

x_i are position vectors.

Barycentric coords

Directed vol. element: $\Delta X = \frac{1}{n!} e_1 \wedge \cdots \wedge e_n$

Point in simplex: $x = x_0 + \sum_{i=1}^{n} \lambda^i e_i$ where $\left.\begin{array}{l} 0 \le \lambda^i \le 1 \\ \text{and } \sum_{i=1}^{n} \lambda^i \le 1 \end{array}\right]$ (2i)

$$\int dX = \int_0^1 \int_0^{1-\lambda^n} \cdots \int_0^{1-\sum_{i=2}^{n}\lambda^i} \underbrace{e_1 \wedge \cdots \wedge e_n}_{J} \, d\lambda^1 \cdots d\lambda^n \qquad A = \sum_{i=3}^{n} \lambda^i$$

$$= \int_0^1 \int_0^{1-\lambda^n} \cdots \underbrace{\int_0^{1-A} 1 - \lambda^2 - A \, d\lambda^2}_{} \cdots d\lambda^n \cdot J$$

$$\to \left[-\frac{1}{2} \left(1 - \lambda^2 - A\right)^2 \right]_0^{1-A} = -\frac{1}{2}\left(0 - (1-A)^2\right) = \frac{1}{2}(1-A)^2$$

note:
λ^i is not λ to the power of i, but the i'th λ.

$$= \int_0^1 \int_0^{1-\lambda^n} \cdots \underbrace{\int_0^{1-B} \frac{1}{2}\left(1 - \lambda^3 - B\right)^2 d\lambda^3}_{} \cdots d\lambda^n \cdot J \qquad B = \sum_{i=4}^{n} \lambda^i$$

$$\to \frac{1}{3!}(1-B)^3$$

$$= \int_0^1 \frac{1}{(n-1)!}\left(1 - \lambda^n\right)^{n-1} d\lambda^n \cdot J = \frac{1}{n!} J = \Delta X \qquad \text{(2ii)}$$

Notation for k-simplex: $\underbrace{(x)_{(k)} = (x_0, x_1, \dots x_k)}$

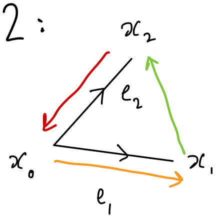
ie just the (ordered) set of (position vectors of) k+1 points defining the simplex.

Boundary of the simplex:

$$\partial (x)_{(k)} = \sum_{i=0}^{k} (-1)^i \underbrace{(x_0, \dots, \check{x}_i, \dots x_k)_{(k-1)}} \qquad (3i)$$

(k-1)-simplex formed by removing x_i

Eg k=2:



$$\partial (x)_{(2)} = (x_1, x_2) - (x_0, x_2) + (x_0, x_1)$$

remove x_0 remove x_1 remove x_2

$(-1)^1$ indicates reversed order

Boundary of a boundary vanishes: $\partial\partial (x)_{(k)} = 0$.

Eg: $\partial\partial (x)_{(2)} = (x_2) - (x_1) - \left[(x_2) - (x_0) \right] + (x_1) - (x_0) = 0$

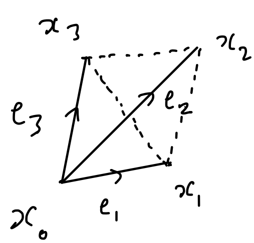
(Directed) volume:

$$\Delta X \equiv \Delta (x)_{(k)} = \frac{1}{k!} e_1 \wedge \dots \wedge \dots e_k = \frac{1}{k!} (x_1 - x_0) \wedge (x_2 - x_0) \wedge \dots \wedge (x_k - x_0)$$

Directed volume of a boundary vanishes:

$$\Delta \left(\partial (x)_{(k)} \right) = 0.$$

To verify, distribute Δ over the sum in the boundary expression. For example, k=3:



$$\partial(x)_{(3)} =$$

$$(x_1, x_2, x_3) - (x_0, x_2, x_3) + (x_0, x_1, x_3) - (x_0, x_1, x_2)$$

$$\Delta\partial(x)_{(3)} = \frac{1}{2!}\Big[$$

$$\underbrace{(x_2-x_1)(x_3-x_1)} - \underbrace{(x_2-x_0)\wedge(x_3-x_0)} + \underbrace{(x_1-x_0)\wedge(x_3-x_0)} - \underbrace{(x_1-x_0)\wedge(x_2-x_0)}\Big]$$

$$\begin{aligned} = \frac{1}{2!}\Big[& \cancel{x_2\wedge x_3} - \cancel{x_2\wedge x_1} - \cancel{x_1\wedge x_3} \\ & - \cancel{x_2\wedge x_3} + \cancel{x_2\wedge x_0} + \cancel{x_0\wedge x_3} \\ & + \cancel{x_1\wedge x_3} - \cancel{x_1\wedge x_0} - \cancel{x_0\wedge x_3} \\ & - \cancel{x_1\wedge x_2} + \cancel{x_1\wedge x_0} + \cancel{x_0\wedge x_2} \Big] = 0 \end{aligned}$$

Note that the directed integral over the boundary of a simplex vanishes:

$$\underset{\partial(x)_{(k)}}{\oint} dS = \sum_{k=0}^{k} (-1)^i \underset{(\check{x}_i)_{(k-1)}}{\int} dX = \Delta\partial(x)_{(k)} = 0 \quad (4i)$$

Distribute integral over the sum in the boundary expression

In a simplex chain, the directed areas of common faces cancel, so can fill a space and only integrate over the space boundary.

Multivector Fields on Simplices

The $\{e_i\}$ are linearly independent, so define a (barycentric) frame, albeit not orthonormal. Can therefore linearly interpolate a multivector field $F(x)$ as $f(x) = F_0 + \sum_{i=1}^{n} \lambda^i (F_i - F_0)$ where $F_i = F(x_i)$. So, can write integral of F over simplex boundary as:

$$\oint_{\partial(x)_{(n)}} f(x)\, dS = \sum_{i=1}^{n} (F_i - F_0) \oint_{\partial(x)_{(n)}} \lambda^i\, dS$$

[F_0 term vanishes because constant - see (4i)]

Now, want to get something like "$\int_{(x)_{(n)}} \nabla F\, dX$" on the RHS, and $F_i - F_0$ is already in the ballpark of a derivative. In fact,

$$F_i - F_0 = \frac{\partial f}{\partial \lambda^i}$$

Given $x = x_0 + \sum_{i=1}^{n} \lambda^i e_i$ from page (2), we have $\lambda^i = e^i \cdot (x - x_0)$ where $\{e^i\}$ is the frame reciprocal to $\{e_i\}$, so that $e^i \cdot e_j = \delta^i_j$.

So we are dealing with something of the form

$$\oint_{\partial(x)_{(n)}} b \cdot x\, dS \quad (5i)$$

where b is e^i and the constant x_0 term vanishes (see (4i))

Because the integral over a simplex boundary is the antisymmetrized sum over simplices one dimension lower, it pays to consider integrals over simplices themselves as on page 2. We already have that

$$\int_{(x)(k)} dX = \Delta X = \frac{1}{k!} e_1\wedge\cdots\wedge e_k \text{, to repeat (2ii).}$$

Can show similarly that $\displaystyle\int_{(x)(k)} \lambda^i\, dX = \frac{1}{k+1}\Delta X \quad \forall\, i$:

$$\int_0^1\int_0^{1-\lambda^k}\cdots\int_0^{1-A^i} \underbrace{\lambda^i}_{v}\, \overbrace{\frac{(1-\lambda^i-A^i)^{i-1}}{(i-1)!}}^{u'}\, d\lambda^i\cdots d\lambda^k \cdot J \qquad A^i=\sum_{j=i+1}^{k}\lambda^j$$

"$\int u'v = uv - \int uv'$"

$$\frac{1}{(i-1)!}\left(\underbrace{\left[\frac{(1-\lambda^i-A^i)^i}{i}\lambda^i\right]_0^{1-A^i}}_{0} - \int_0^{1-A^i}\frac{(1-\lambda^i-A^i)^i}{i}\,d\lambda^i\right) = \frac{(1+A^i)^{i+1}}{(i+1)!}$$

The λ^i in the i'th integral increments the power and factorial by an extra 1, which gives at the end

$$\int_{(x)(k)} \lambda^i\, dX = \frac{1}{(k+1)!}J = \frac{1}{k+1}\Delta X \qquad (6i)$$

From (5i), focus on $b \cdot x$. In barycentric coordinates (2i), 7

$$b \cdot x = b \cdot x_0 + \sum_{i=1}^{k} b_i \lambda^i \quad \text{where} \quad b_i = b \cdot e_i$$

So
$$\int_{(x)_{(k)}} b \cdot x \, dX = b \cdot x_0 \underbrace{\Delta X}_{(2ii)} + \sum_{i=1}^{k} b_i \underbrace{\int_{(x)_{(k)}} \lambda^i \, dX}_{(6i)}$$

$$= \underbrace{\left(b \cdot x_0 + \frac{1}{k+1} \sum_{i=1}^{k} b_i \right)} \Delta X$$

geometric center of simplex, $b \cdot \bar{x}$, because:

$$\sum_{i=1}^{k} b_i = \sum_{i=1}^{k} b \cdot (x_i - x_0) = \sum_{i=1}^{k} b \cdot x_i - k \, b \cdot x_0$$

So
$$b \cdot x_0 + \frac{1}{k+1} \sum_{i=1}^{k} b_i = \frac{1}{k+1} \left((\cancel{k}+1) \, b \cdot x_0 \cancel{- k \, b \cdot x_0} + \sum_{i=1}^{k} b \cdot x_i \right)$$

$$= \frac{1}{k+1} \sum_{i=0}^{k} b \cdot x_i$$

note

So we have that:

$$\int_{(x)_{(k)}} b \cdot x \, dX = b \cdot \bar{x} \, \Delta X \qquad (7i)$$

However, (5i) integrates over $\partial (x)_{(k)}$ (the boundary), not $(x)_{(k)}$ (the simplex), so let's address that next.

From (3_i),

$$\oint_{\partial(x)_{(k)}} b\cdot x\, dS = \sum_{i=0}^{k}(-1)^i \int_{(\check{x}_i)_{(k-1)}} b\cdot x\, dX$$

where $(\check{x}_i)_{(k-1)}$ is the $k-1$ simplex formed by omitting x_i from $(x)_{(k)}$.

$$= \sum_{i=0}^{k}(-1)^i \frac{1}{k} b\cdot\left(x_0 + \cdots \check{x}_i + \cdots x_k\right)\Delta(\check{x}_i)_{(k-1)} \text{ by } (7_i)$$

$$= \frac{b\cdot X}{k}\underbrace{\sum_{i=0}^{k}(-1)^i\Delta(\check{x}_i)_{(k-1)}}_{=0 \text{ by } (4_i)} + \frac{1}{k}\sum_{i=0}^{k}(-1)^i b\cdot x_i\,\Delta(\check{x}_i)_{(k-1)}$$

where $X = \sum_{i=0}^{k} x_i$

$$= \frac{1}{k}\sum_{i=0}^{k}(-1)^i b\cdot(\underbrace{x_i - x_0}_{})\,\underbrace{\Delta(\check{x}_i)_{(k-1)}}_{}$$

constant so can add in by (4_i)

$$= \frac{1}{k}\sum_{i=0}^{k}(-1)^i b\cdot e_i\, \frac{1}{(k-1)!} e_1\wedge\cdots\check{e}_i\cdots\wedge e_k$$

using the identity:

$$b\cdot(a_1\wedge\cdots\wedge a_r) = \sum_{i=1}^{r}(-1)^{i+1} b\cdot a_i\,(a_1\wedge\cdots\check{a}_i\cdots a_r)$$

$$= \frac{1}{k!} b\cdot\left(e_1\wedge\cdots\wedge e_k\right)$$

So,

$$\oint_{\partial(x)_{(k)}} b\cdot x\, dS = b\cdot\Delta X \qquad (8_i)$$

This is the key result that gives us the FTGC applied to page 5.

On page 5 we had:

$$\oint_{\partial(x)_{(n)}} f(x)\, dS = \sum_{i=1}^{n} (F_i - F_0) \oint_{\partial(x)_{(n)}} \lambda^i \, dS$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda^i} \oint_{\partial(x)_{(n)}} e^i \cdot (x - \cancel{x_0})\, dS$$

0 by (4i)

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial \lambda^i}\, e_i \cdot \Delta X \quad \text{by (8i)}$$

$$= \dot{f}\, \dot{\nabla} \cdot \Delta X \quad \text{where} \quad \nabla = \sum_i e^i \partial_i$$

in barycentric coords.

Now, all of this is for one simplex. By filling a space V with a simplex chain $\{(x)^a_{(n)}\}$ and taking the limit of vanishing simplex size, we get

$$\oint_{\partial V} F\, dS = \int_V \dot{F}\, \dot{\nabla}\, dX \qquad (9i)$$

$\left(\lim \sum_a \dot{f}\, \dot{\nabla} \cdot \Delta X^a\right)$

note: inner product removed because outer product contributes zero because grade > dims of V.

Rework the above with multivector field to right of measure:

$$\oint_{\partial V} dS\, G = \int_V \dot{\nabla}\, dX\, \dot{G} \qquad (9ii)$$

Can generalize further by defining a linear function $L(A_{n-1};x)$ consuming a grade $n-1$ blade and returning a multivector of any grades. Linear interpolation:

$$L(A;x) = L(A;x_0) + \sum_{i=1}^{n} \lambda^i \big(L(A;x_i) - L(A;x_0) \big)$$

Linearity means commutation with all sums / integrals, and we get:

$$\oint_{\partial V} L(dS) = \int_V \dot{L}(\dot{\nabla} dX) \qquad (10;)$$

(position-dependence suppressed)

See next page (10a) for summary of proof

Applications

The Divergence Theorem follows from (10;) with:

(n-dim) $\qquad L(A) = \langle J A I^{-1} \rangle$ for a vector field J.

$\langle J\, dS\, I^{-1} \rangle$

grades 1 $\quad n-1 \quad n$

$= \langle J\, n\, |dS| \rangle$

$= n \cdot J\, |dS|$

where $n|dS| = dS I^{-1}$

$$\langle \underbrace{\dot{J}\, \dot{\nabla}}_{\text{scalar}}\, \underbrace{dX\, I^{-1}}_{|dX| = I^{-1} dX} \rangle$$

grades 1 1 n n

scalar $\rightarrow$ scalar $\nabla \cdot J$ (anticommuting term vanishes as grade 2)

$|dX| = I^{-1} dX$

can cycle multivectors in $\langle\ \rangle$

$$\therefore \oint_{\partial V} n \cdot J\, |dS| = \int_V \nabla \cdot J\, |dX|$$

Interlude: Summary of Main Steps in Proof

Linearly interpolate in barycentric coords

$$f(x) = F_0 + \sum_{i=1}^{n} \lambda^i (F_i - F_0)$$

$$e^i \cdot (x - x_0)$$

Integral over simplex boundary then focuses on difference $F_i - F_0$ and the simplex geometry

$$\oint_{\partial(x)_{(n)}} f(x)\, dS = \sum_{i=1}^{n} (F_i - F_0) \oint_{\partial(x)_{(n)}} \lambda^i\, dS$$

Integral of inner product over boundary becomes inner product with directed volume,

$$\oint_{\partial(x)_{(k)}} b \cdot x\, dS = b \cdot \Delta X \qquad (8i)$$

because integral of inner product over directed volume ie inner product with geometric center of simplex,

$$\int_{(x)_{(k)}} b \cdot x\, dX = b \cdot \bar{x}\, \Delta X \qquad (7i)$$

because integral of barycentric coordinate picks up factorial increment which fixes normalization.

$$\int_{(x)_{(k)}} \lambda^i\, dX = \frac{1}{(k+1)!} J = \frac{1}{k+1} \Delta X \qquad (6i)$$

Then, turn difference in linear interpolation into a derivative and take the limit of the sum over simplices in a chain filling region V

$$\dot{f} \dot{\nabla} \cdot \Delta X \quad \text{where} \quad \nabla = \sum_i e^i \partial_i$$

$$\oint_{\partial V} F\, dS = \int_V \dot{F} \dot{\nabla}\, dX$$

Green's Theorem follows from (9ii) with G being the 2-d vector field $J = Pe_1 + Qe_2$ $\quad (e_i \cdot e_j = \delta_{ij})$

$$\oint_{\partial V} dS\, J = \int_V \dot{\nabla} \underbrace{\underset{2}{dX}\ \underset{1}{\dot{J}}}_{\text{grades}} = -\int_V \nabla J\, dX$$

vector J anticommutes with 2-d pseudoscalar $dX = I\,dx\,dy$

$$\nabla J = (\partial_x e_1 + \partial_y e_2)(Pe_1 + Qe_2)$$

$$= \partial_x P + \partial_y Q + (\partial_x Q - \partial_y P)\, e_1 e_2$$

LHS is a scalar. Scalar part of $\nabla J I = -(\partial_x Q - \partial_y P)$

$$\therefore \oint P\,dx + Q\,dy = \int (\partial_x Q - \partial_y P)\, dx\,dy.$$

Stokes' Theorem arises when $L(a) = J \cdot a$ for vectors J and a.

So (10i) becomes

$\wedge$ term is grade 3 but LHS is scalar

$$\oint_{\partial V} J \cdot dL = \int_V \underset{1}{J} \cdot (\underset{1}{\nabla} \cdot \underset{2}{dX})$$

$$= -\int_V (\nabla \wedge J) \cdot dX$$

because $a \cdot (b \cdot B) = (a \wedge b) \cdot B$ for vectors a, b and bivector B.

For Cauchy's integral formula, need the notion of an analytic complex function, or more generally a monogenic multivector field.

* also a multivector field

Let $\varphi = u e_1 + v e_2$ and $\psi = e_1 \varphi = u + vI$ be a 2-d vector field and the corresponding complex function*, which is analytic when the Cauchy-Riemann conditions are met:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad \text{or} \quad u_x = v_y \text{ and } u_y = -v_x .$$

This means the map defined by the complex function is locally conformal. Small changes are well-approximated by the Jacobian

$$\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{given } \begin{array}{l} u_x = a \\ v_x = b \end{array} \text{ and CR.}$$

$$= \underbrace{w \underbrace{\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}}_{\text{rotation}}}_{\text{"magnitwist" only - no shearing.}} \quad \text{for } \begin{array}{l} w^2 = a^2 + b^2 \\ \tan\phi = \frac{b}{a} \end{array}$$

uniform dilation

"magnitwist" only - no shearing.
Squares can get bigger/smaller and change angle, but do not become parallelograms.

This was Joseph's insight.

Monogenic functions satisfy $\nabla\psi = 0$:

$$\nabla\psi = (\partial_x e_1 + \partial_y e_2)(u + v e_1 e_2)$$

$$= \underbrace{(u_x - v_y) e_1 + (u_y + v_x)}_{\text{Cauchy-Riemann}} e_2 = 0$$

The vector derivative of the corresponding vector field shows the magnituist explicitly:

$$\nabla\varphi = \nabla\cdot\varphi + \nabla\wedge\varphi$$

$$= (\partial_x e_1 + \partial_y e_2)(u e_1 + v e_2)$$

$$= (u_x + v_y) + (v_x - u_y) e_1 e_2 = 2a + 2bI = 2we^{I\phi}.$$

The FTGC in the form of (9ii) with $G = \psi$ is

$$\oint_{\partial V} \frac{\partial r}{\partial \lambda} \psi \, d\lambda = \int_V \nabla\psi \, dX$$

ψ commutes with dX because even-grade.

$r = x e_1 + y e_2$ curve param

Map to complex plane by premultiplying by e_1 :

$$e_1 \oint_{\partial V} \frac{\partial r}{\partial \lambda} \psi \, d\lambda = \oint_{\partial V} \frac{\partial z}{\partial \lambda} \psi \, d\lambda = \oint \psi \, dz = \int e_1 \underbrace{\nabla\psi}_{=0 \text{ if } \psi \text{ monogenic}} dX \quad (13i)$$

$\therefore$ have Cauchy's Theorem: $\oint \psi \, dz = 0$ for analytic ψ.

Cauchy's integral formula (CIF) is:

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\, dz \quad \text{for analytic } f(z).$$

Now: $$\frac{1}{z-a} = \frac{(z-a)^\dagger}{|z-a|^2} = \frac{(r-b)e_1}{(r-b)^2} \quad \text{as } z^\dagger = (e_1 r)^\dagger = re_1, \quad a^\dagger = be_1$$

and $\frac{r-b}{(r-b)^2} = \nabla \ln|r-b|$. We also know that $\ln|r-b|$ is the Green function for ∇^2 in 2-d: $\nabla^2 \ln|r-b| = 2\pi\delta(r-b)$

so $$\underline{\nabla \frac{r-b}{(r-b)^2} = \nabla^2 \ln|r-b| = 2\pi\delta(r-b)}$$

existence of Green fn $\Updownarrow$ $\underline{\nabla \text{ invertible}}$.

Now let $\psi = \frac{f(z)}{z-a}$ in (13i):

$g(r) = g(e_1 z) = f(z)$

$$\oint \frac{f(z)}{z-a}\, dz = e_1 \int \nabla\left(\frac{r-b}{(r-b)^2} e_1 g(r)\right) \underbrace{dX}_{I|dX|}$$

$$= e_1 \int \left(2\pi\delta(r-b)\, e_1 g(r) + \cancel{\nabla f(r)} \frac{r-b}{(r-b)^2} e_1\right) I|dX|$$

$\cancel{\nabla f(r)} \to 0$ for analytic f.

$$= 2\pi I f(a)$$

If f not analytic, now we have generalized Cauchy's result!

$\frac{1}{z-a}$ is the Green function for the vector derivative in 2-d, and C.I.F. propagates $f(z)$ off the boundary to any point a.